Chapter 4 Series Solutions

- Explicit and closed formed solution of a differential equation

  $y' + 2y = 1; \qquad y(0) = 3$

  $y(x) = \frac{1}{2}(1 + 5e^{-2x})$

  Closed form: since finite algebraic combination of elementary functions

- Series solution: giving $y(x)$ as an infinite series involving constant time powers of $x$

(Def 4.4) Power Series

$$\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n \text{ converges for } x = x_0$$

Ex62: $\sum_{n=0}^{\infty} n!x^n$, which converges for $x = 0$ but diverges for any $x \neq 0$

(Thm 4.5)

Suppose $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ converges for $x = x_1 \neq x_0$. Then the series converges absolutely for all $x$ such that

$$\left|x - x_0\right| < \left|x_1 - x_0\right|$$

Three possibilities for convergence of $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ :

1. converge only for $x = x_0$
2. converge for all real number $x$
3. positive number $r$, called the radius of convergence of the series, such that $\sum_{n=0}^{\infty} a_n \left(x - x_0\right)^n$ converges if $\left|x - x_0\right| < r$, diverges if $\left|x - x_0\right| > r$. $\left(x_0 - r, x_0 + r\right)$ is called the open interval of convergence.

(Thm 4.6)

Suppose $b_n \neq 0$ for $n = 0,1,2,...$ and that

$$\lim_{n\to\infty}\left|\frac{b_{n+1}}{b_n}\right| = L$$

Then $\sum_{n=0}^{\infty} b_n$ converges absolutely if $L < 1$ and diverges if $L > 1$.

If $L = 1$, then this test allows no conclusion

◆ Taylor and Maclaurin Expansions

$$f(x)=\sum_{n=0}^{\infty}a_n\left(x-x_0\right)^n \quad \text{in some interval } \left(x_0-r,x_0+r\right)$$

$$f\left(x_0\right)=a_0$$

$$f'\left(x_0\right)=a_1$$

$$f''\left(x_0\right)=2a_2$$

$$f^{(3)}\left(x_0\right)=3\bullet 2a_3$$

$$f^{(k)}\left(x_0\right)=k\left(k-1\right)\cdots\left(2\right)\left(1\right)a_k$$

$$a_k=\frac{1}{k!}f^{(k)}(x_0)$$

This number is called the $k$th Taylor coefficient of $f$ at $x_0$

$$\sum_{n=0}^{\infty}\frac{1}{n!}f^{(n)}\left(x_0\right)\left(x-x_0\right)^n$$

The series is the Taylor series for $f(x)$ about $x_0$

If $x_0=0$, the Taylor expansion is about 0 and is often called a Maclaurin expansion

$$e^x=\sum_{n=0}^{\infty}\frac{1}{n!}x^n$$

$$\sin\left(x\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^n}{\left(2n+1\right)!}x^{2n+1}$$

are Maclaurin expansions

4.1 Power Series Solutions of Initial Value Problems

(Def 4.1) Analytic Function:

A function $f$ is analytic at $x_0$ if $f(x)$ has a power series representation in some open interval about $x_0$:

$$f(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

in some interval $(x_0-r, x_0+r)$. r is the radius of convergence (收斂半徑）

(Thm 4.1)

Let $p$ and $q$ be analytic at $x_0$. Then the initial value problem

$$y'+p(x)y=q(x); \qquad y(x_0)=y_0$$

has a solution that is analytic at $x_0$

That means coefficients are analytic at $x_0$ has an analytic solution at $x_0$

$$y(x)=\sum_{n=0}^{\infty}a_n(x-x_0)^n$$

$$a_n=\frac{1}{n!}y^{(n)}(x_0).$$

Ex63: $y' + e^{x} y = x^{2}$; $y(0) = 4$

[解]:

(Thm 4.2)

Let $p$, $q$, and $f$ be analytic at $x_0$. Then the initial value problem

$$y''+p(x)y'+q(x)y=f(x); \quad y(x_0)=A, \; y'(x_0)=B$$

has a unique solution that is also analytic at $x_0$

Ex64: $y''-xy'+e^{x}y=4$; $y(0)=1$, $y'(0)=4$

[解]:

Exercise V:

In each problem, find the first five nonzero terms of the power series solution of the initial value problem, about the point where the initial conditions are given.

1. $y''+y'-xy=0$; $y(0)=-2$, $y'(0)=0$

$\{\, y=-2-\frac{1}{3}x^3+\frac{1}{12}x^4-\frac{1}{60}x^5-\frac{1}{120}x^6+\cdots \}$

2. $y''-xy=2x$; $y(1)=3$, $y'(1)=0$

$\{\, y=3+\frac{5}{2}(x-1)^2+\frac{5}{6}(x-1)^3+\frac{5}{24}(x-1)^4+\frac{1}{6}(x-1)^5+\cdots \}$

3. $y''-e^x y'+2y=1$; $y(0)=-3$, $y'(0)=1$

$\{\, y=-3+x+4x^2+\frac{7}{6}x^3+\frac{1}{3}x^4+\cdots \}$

4.2 Power Series Solutions Using Recurrence Relations

Ex65: $y''+x^2y=0$ ，we want a solution expanded about 0

[解]: $y\left(x\right)=\sum_{n=0}^{\infty}a_nx^n$

$$y'=\sum_{n=1}^{\infty}na_nx^{n-1}$$

$$y''=\sum_{n=2}^{\infty}n\left(n-1\right)a_nx^{n-2}$$

$y'$ begins at $n=1$ ，$y''$ begins at $n=2$

Substitute $y, y''$ into the differential equation

$$y''+x^2y=\sum_{n=2}^{\infty}n\left(n-1\right)a_nx^{n-2}+\sum_{n=0}^{\infty}a_nx^{n+2}=0$$

$$\sum_{n=2}^{\infty}n\left(n-1\right)a_nx^{n-2}=\sum_{m=0}^{\infty}\left(m+2\right)\left(m+1\right)a_{m+2}x^m$$

$$(m=n-2, n=m+2)$$

$$\sum_{n=0}^{\infty}a_nx^{n+2}=\sum_{m=2}^{\infty}a_{m-2}x^m$$

$$(m=n+2, n=m-2)$$

$$(m=n)$$

$$\sum_{n=0}^{\infty}\left(n+2\right)\left(n+1\right)a_{n+2}x^n+\sum_{n=2}^{\infty}a_{n-2}x^n=0$$

$$2\left(1\right)a_2x^0+3\left(2\right)a_3x+\sum_{n=2}^{\infty}\left[\left(n+2\right)\left(n+1\right)a_{n+2}+a_{n-2}\right]x^n=0$$

$$a_2 = a_3 = 0$$

$$(n+2)(n+1)a_{n+2} + a_{n-2} = 0 \quad \text{for n=2, 3, ….}$$

$$a_{n+2} = -\frac{1}{(n+2)(n+1)}a_{n-2} \quad \text{for n = 2, 3,….}$$

$$a_4 = -\frac{1}{4(3)}a_0 = -\frac{1}{12}a_0$$

$$a_5 = -\frac{1}{5(4)}a_1 = -\frac{1}{20}a_1$$

$$y(x) = a_0 + a_1 x + 0x^2 + 0x^3 - \frac{1}{12}a_0 x^4$$

$$-\frac{1}{20}a_1 x^5 + 0x^6 + 0x^7 + \frac{1}{672}a_0 x^8 + \frac{1}{1440}a_1 x^9 + \cdots$$

$$= a_0\left(1 - \frac{1}{12}x^4 + \frac{1}{672}x^8 + \cdots\right) + a_1\left(x - \frac{1}{20}x^5 + \frac{1}{1440}x^9 + \cdots\right)$$

$$a_0 = y(0) \text{，} a_1 = y'(0)$$

Ex66： $y''+x^2y'+4y=1-x^2$

[解]:

Exercise W:

In each problem, find the recurrence relation and use it to generate the first five terms of the Maclaurin series of the general solution.

1. $y' - xy = 1 - x$

$$\{\, y = 1 + \sum_{n=0}^{\infty} \frac{1}{1\cdot 3\cdot 5\cdots(2n+1)} x^{2n+1} + c_0\left(1 + \sum_{n=1}^{\infty} \frac{1}{2\cdot 4\cdot 6\cdots 2n} x^{2n}\right),$$

$c_0 = a_0 - 1\}$

2. $y'' + 2y' + xy = 0$

$\{\quad y = a_0 y_1 + a_1 y_2, a_0 = y(0), a_1 = y'(0)$

$$y_1 = 1 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{30}x^5 + \frac{1}{60}x^6 + \cdots$$

$$y_2 = x - x^2 + \frac{2}{3}x^3 - \frac{5}{12}x^4 + \frac{13}{60}x^5 + \cdots\}$$

3. $y'' + y' - (1 - x + x^2)y = -5$

$$\{\quad y = a_0\left[1 + \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4 - \frac{1}{12}x^5 + \cdots\right] +$$

$$+a_1\left[x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{5}{24}x^4 + \frac{2}{15}x^5 - \cdots\right]$$

$$-5\left[\frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{20}x^5 + \frac{1}{30}x^6 - \cdots\right] \quad a_0 = y(0), a_1 = y'(0)\}$$

## 4.3 Singular Points and the Method of Forbenius

$$P(x)y''+Q(x)y'+R(x)y=F(x)$$

$$y''+p(x)y'+q(x)y=f(x)$$

(Def 4.2) Ordinary and Singular Points:

$x_0$ is an ordinary point of equation

$P(x)y''+Q(x)y'+R(x)y=F(x)$ if $P(x_0)\neq 0$ and

$Q(x)/P(x)$，$R(x)/P(x)$，and $F(x)/P(x)$ are analytic at $x_0$

$x_0$ is a singular point of equation

$P(x)y''+Q(x)y'+R(x)y=F(x)$ if $x_0$ is not an ordinary

point

Ex67: $x^3(x-2)^2y''+5(x+2)(x+2)y'+3x^2y=0$

[解]:

◆ Only focus on homogeneous equation

$$P(x)y''+Q(x)y'+R(x)y=0$$

(Def 4.3) Regular and Irregular Singular Points:

$x_0$ is a regular singular point of equation

$P(x)y''+Q(x)y'+R(x)y=0$ if $x_0$ is a singular point, and the functions $(x-x_0)\dfrac{Q(x)}{P(x)}$ and $(x-x_0)^2\dfrac{R(x)}{P(x)}$ are analytic at $x_0$. A singular point that is not regular is said to be an irregular singular point

Ex68: $x^3(x-2)^2y''+5(x+2)(x-2)y'+3x^2y=0$

[解]:

For regular singular point at $x_0$

$$y(x)=\sum_{n=0}^{\infty}c_n\left(x-x_0\right)^{n+r}$$ is called Forbenius series

A Forbenius series need not be a power series, r may be negative or a noninteger

“Begins” with $c_0(x-x_0)^r$, which is constant only if $r=0$

$$y'(x)=\sum_{n=0}^{\infty}(n+r)c_n\left(x-x_0\right)^{n+r-1}$$

This summation for $y'$ begins at zero. Also true for $y''$

$$y''(x)=\sum_{n=0}^{\infty}(n+r)(n+r-1)c_n\left(x-x_0\right)^{n+r-2}$$

Ex69: $x^2y''+x\left(\frac{1}{2}+2x\right)y'+\left(x-\frac{1}{2}\right)y=0$

[解]:

## 4.4 Second Solutions

(Thm 4.4) A second solution in the method of Frobenius:

If $(r_1 - r_2)$ is not an integer, two linearly independent Frobenius

solutions:

$$y_1(x) = \sum_{n=0}^{\infty} c_n x^{n+r_1} \quad \text{and} \quad y_2(x) = \sum_{n=0}^{\infty} c_n^* x^{n+r_2} \qquad c_0 \neq 0, \ \ c_0^* \neq 0$$

$y_1$ and $y_2$ form a fundamental set on some interval (0, r) or (-r,

0)

Exercise X:

Find all of the singular points and classify each singular point as regular or irregular.

1. $x^2\left(x-3\right)^2 y''+4x\left(x^2-x-6\right)y'+\left(x^2-x-2\right)y=0$

{ Singular points are $x=0$ and $x=3$.

$x=0$ is regular, $x=3$ is regular}

In each problem, (a) show that zero is a regular singular point of the differential equation, (b) find and solve the indicial equation, (c) determine the recurrence relation and (d) use the results of (b) and (c) to find the first five nonzero terms of two linearly independent Frobenius solutions.

2. $4x^2y''+2xy'-xy=0$

{(a) $\dfrac{xQ\left(x\right)}{P\left(x\right)}=\dfrac{1}{2}$ and $x^2\dfrac{R\left(x\right)}{P\left(x\right)}=-\dfrac{x}{4}$ are both analytic at $x=0$ so, $x=0$ is a regular singular point.

(b) $2r\left(2r-1\right)=0$ with roots $r_1=\dfrac{1}{2}$ and $r_2=0$

(c) The recurrence relation is

$$c_n=\frac{1}{2\left(n+r\right)\left(2\left(n+r\right)-1\right)}c_{n-1}, n\ge 1$$

(d) Using $r=\frac{1}{2}$ gives $c_n=\frac{1}{2n(2n+1)}c_{n-1}$ and one solution

$$y_1=x^{1/2}\left[1+\sum_{n=1}^{\infty}\frac{x^n}{(2n+1)!}\right]$$

Using $r=0$ gives $c_n=\frac{1}{(2n-1)(2n)}c_{n-1}$ and a second

solution $y_2=1+\sum_{n=1}^{\infty}\frac{x^n}{(2n)!}$ }

3. $9x^2y''+2(2x+1)y=0$

{(a) $\frac{xQ(x)}{P(x)}=0$ and $\frac{x^2R(x)}{P(x)}=\frac{2(2x+1)}{9}$, both analytic at $x=0$

(b) $9r^2-9r+2=0$ with roots $r_1=2/3$ and $r_2=1/3$

(c) $c_n=\frac{-4}{(3(n+r)-2)(3(n+r)-1)}c_{n-1}, n\geq 1$

(d) Using $r=2/3$ gives $c_n=\frac{-4}{3n(3n+1)}c_{n-1}$ and

$$y_1=x^{2/3}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{3^n n!\left[4\cdot 7\cdot 10\cdots(3n+1)\right]}x^n\right]$$

Whereas $r=1/3$ gives $c_n=\frac{-4}{(3n-1)3n}c_{n-1}$ and

$$y_2=x^{1/3}\left[1+\sum_{n=1}^{\infty}\frac{(-1)^n 4^n}{3^n n!\left[2\cdot 5\cdot 8\cdots(3n-1)\right]}x^n\right]\}$$